Eigenvalue problem of Sturm-Liouville systems with separated boundary conditions

Xijun Hu* Penghui Wang†

Department of Mathematics, Shandong University
Jinan, Shandong 250100, The People's Republic of China

Abstract

Let λ_j be the j-th eigenvalue of Sturm-Liouville systems with separated boundary conditions, we build up the Hill-type formula, which represent $\prod_j(1-\lambda_j^{-1})$ as a determinant of finite matrix. This is the first attack on such a formula under non-periodic type boundary conditions. Consequently, we get the Krein-type trace formula based on the Hill-type formula, which express $\sum_j \frac{1}{\lambda_j^m}$ as trace of finite matrices. The trace formula can be used to estimate the conjugate point alone a geodesic in Riemannian manifold and to get some infinite sum identities.

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Key Words. Hill-type formula, trace formula, Hamiltonian systems, Sturm-Liouville systems

1 Introduction

In this paper, we will consider the eigenvalue problem for the Sturm-Liouville systems

$$-(P\dot{y}+Qy)^{\cdot}+Q^T\dot{y}+(R+\lambda R_1)y=0, \tag{1.1}$$

where Q is a continuous path of $n\times n$ matrices, and P, R, R_1 are continuous paths of $n\times n$ symmetric matrices on $[0,T]$. Instead of Legendre convexity condition, we assume that for any $t\in[0,T]$, $P(t)$ is invertible.

The eigenvalue problem of the Sturm-Liouville systems depends on the boundary conditions. There are two important type boundary conditions, periodic type and separated type. For the literature in n-body problem, readers can refer to [5],[8]. The eigenvalue problem for S-periodic boundary value problem, that is, $y(0)=Sy(T)$ for some orthogonal matrix S, was studied in [7]. In the present paper, we will consider the separated boundary conditions, which includes the homogenous Dirichlet, Neumann and Robin boundary conditions. More precisely, let Λ_0,Λ_1 be two Lagrangian subspaces of $(\mathbb{R}^{2n},\omega_0)$ which are phase spaces with standard symplectic structure. Set $x=P\dot{y}+Qy$, $z=(x,y)^T$, and the separated boundary condition is given by

$$z(0)\in\Lambda_0, \quad z(T)\in\Lambda_1. \tag{1.2}$$

*Partially supported by NSFC(No.11425105, 11131004), NCET, E-mail:xjhu@sdu.edu.cn
†Partially supported by NSFC(No.11471189), E-mail: phwang@sdu.edu.cn

In order to understand the eigenvalue problem of the system (1.1-1.2), for the eigenvalues λ_j we will build up a formula with the form $\prod_j(1-\lambda_j^{-1}) = \det(\mathcal{M})$, where the matrix $\mathcal{M}$ depends mainly on the monodromy matrix and the Lagrangian subspaces Λ_0, Λ_1. We called it Hill-type formula because a similar formula for periodic orbits was shown by Hill when he considered the motion of lunar perigee [6] at 1877. However, Hill did not prove the convergence of the infinite determinant, and the convergence was given by Poincaré [15]. Afterwards, the Hill-type formula for a periodic solution of Lagrangian system on manifold was given by Bolotin [1]. For more results, please refer ([2],[4],[10],[3]) etc. We should point out that, till now, all the known results on the Hill-type formula were given for the periodic-type boundary problem.

To state Hill-type formula for the separated boundary conditions, we firstly introduce some notations. Suppose Λ is a Lagrangian subspace of $(\mathbb{R}^{2n}, \omega_0)$, a Lagrangian frame for Λ is a linear map $Z : \mathbb{R}^n \to \mathbb{R}^{2n}$ whose image is Λ. It is easy to see that the frame is of the form $Z = \begin{pmatrix} X \\ Y \end{pmatrix}$, where X, Y are $n \times n$ matrices and satisfied $X^T Y = Y^T X$.

By the standard Legendre transformation, the linear system (1.1) with the boundary conditions (1.2) corresponds to the linear Hamiltonian system,

$$\dot{z} = JB_\lambda(t)z, \quad z(0) \in \Lambda_0, \quad z(T) \in \Lambda_1, \tag{1.3}$$

with

$$B_\lambda(t) = \begin{pmatrix} P^{-1}(t) & -P^{-1}Q(t) \\ -Q(t)^T P^{-1}(t) & Q(t)^T P^{-1}(t)Q(t) - R(t) - \lambda R_1(t) \end{pmatrix}. \tag{1.4}$$

Without confusion, for Lagrangian system, denote $\gamma_\lambda(t)$ the fundamental solution of (1.3), that is $\dot{\gamma}_\lambda(t) = JB_\lambda(t)\gamma_\lambda(t)$ with $\gamma_\lambda(0) = I_{2n}$. Let Z_0, Z_1 be frames of Λ_0, Λ_1. Obviously, $\gamma_\lambda(T)Z_0$ are frames of $\gamma_\lambda(T)\Lambda_0$. $(\gamma_\lambda(T)Z_0, Z_1)$ are $2n \times 2n$ matrices.

To simplify the notation, set $A = -\frac{d}{dt}(P\frac{d}{dt} + Q) + Q^T\frac{d}{dt} + R$, which is a self-adjoint operator on $L^2([0,T], \mathbb{R}^n)$ with domain:

$$D(\Lambda_0, \Lambda_1) = \{y \in W^{2,2}([0,T], \mathbb{R}^n), z(0) \in \Lambda_0, z(T) \in \Lambda_1\}.$$

Throughout of the paper, without loss of generality, we will assume A is nondegenerate, that is, 0 is not an eigenvalue of (1.1-1.2). It is obvious that λ is a nonzero eigenvalue of the system(1.1-1.2) if and only if $-\frac{1}{\lambda}$ is an eigenvalue of R_1A^{-1}. In what follows, the multiplicity of an eigenvalue λ_j means the algebraic multiplicity of R_1A^{-1} at $-1/\lambda_j$.

Theorem 1.1. *Under the nondegenerate assumption, we have*

$$\prod_j(1-\lambda_j^{-1}) = \det(\gamma_1(T)Z_0, Z_1) \cdot \det(\gamma_0(T)Z_0, Z_1)^{-1}, \tag{1.5}$$

where the left infinite product takes on the eigenvalues λ_j counting the multiplicity.

Remark 1.2. *The Hill-type formula for S-periodic orbits was built up in [7] with the following form*

$$\prod_j(1-\lambda_j^{-1}) = \det(\gamma_1(T) - S_d) \cdot \det(\gamma_0(T) - S_d), \tag{1.6}$$

where $S_d = diag(S, S)$. Although (1.5) is similar to (1.6), the proof is different. (1.6) is derived from the Hill-type formula for S-periodic orbits of Hamiltonian systems [9]. The proof of (1.5) is direct, and could

cover the case of (1.6). To the best of our knowledge, (1.5) is the first study on the Hill-type formula of non-periodic type boundary problem. The corresponding formula in Hamiltonian systems for the orbits with Lagrangian boundary conditions is still open.

Trace formula is a powerful tool in the study of eigenvalue problem, especially in estimating the first eigenvalue. The first work on the trace formula was established by Krein[13, 14] for the -1-periodic orbits in the simple case. For the system with S-periodic boundary condition, the trace formula was established in [7, 10]. The present paper is a continuous work of [7, 10], and we will build up the trace formula for separated boundary value problem of Sturm-Liouville system. The idea to get the trace formula is similar to that in [7]. That is, using λR_1 instead of R_1, and give Taylor expansion on both sides of the Hill-type formula. With the notations defined in Section 3, we have the following theorem.

Theorem 1.3. *Assume A is non-degenerate, λ_j are eigenvalues of the Sturm-Liouville system (1.1-1.2) counting the multiplicity, we have for any positive integer m,*

$$\sum_j \frac{1}{\lambda_j^m} = m \sum_{k=1}^{m} \frac{(-1)^k}{k} \Big[\sum_{j_1+\cdots+j_k=m} Tr(G_{j_1}\cdots G_{j_k}) \Big], \tag{1.7}$$

where $G_k, k \in \mathbb{N}$, defined in (3.4), are $n \times n$ matrices.

For applications, a main observation is that the trace formula can be used to estimate the non-degeneracy of the system. Moreover, we can estimate the relative Morse index and Maslov-type index. The Maslov-type index is a powerful tool in study the stability problem, please refer [12],[11],[8] for the detail. By using the trace formula and Maslov-type index theory, in [7] we studied the stability region and hyperbolic region of elliptic Lagrangian orbits in planar three body problem.

It is not hard to see that all the results on the applications of trace formula in [7] have twins in the case of separated boundary conditions. We will not list all the theorems here, but give some computations for the trace formula some special case. Let R be a continuous path of $n \times n$ symmetric matrices on $[0, T]$, we consider the system

$$\ddot{y} + \lambda R y = 0. \tag{1.8}$$

Set $R^+ = \frac{1}{2}(R + |R|)$, which is a path of nonnegative symmetric matrices, we have the following corollary.

Corollary 1.4. *Suppose $Tr\Big(\int_0^T \big(t - \frac{t^2}{T}\big) R^+ dt\Big) < 1$, the Dirichlet problem for (1.8) has no nontrivial solution.*

This result can be used to estimate the conjugate point alone a geodesics in Riemannian manifold. For reader's convenience, we give details here. Let $c : [0, a] \to \mathcal{M}$ be a geodesic of Riemannian manifold $\mathcal{M}$. Choose $\{e_1(0), ..., e_n(0)\}$ to be an orthogonal normal basis of $\dot{c}(0)^{\perp} \subset T_{c(0)}\mathcal{M}$. Its parallel transport $\{e_1(t), ..., e_n(t)\}(t \in [0, a])$ along c gives an orthogonal normal basis of $\dot{c}(t)^{\perp} \subset T_{c(t)}\mathcal{M}$. Recall that the Jacobi equation is

$$\frac{D^2 J}{dt} + R(\dot{c}, J(t))\dot{c} = 0.$$

The point $c(t_0)$ is said to be conjugate to $c(0)$ along c, $t_0 \in [0, a]$, if there exists a nontrivial Jacobi field J along c, with $J(0) = J(t_0) = 0$. Suppose $J(t) = \sum_{i=1}^n J_i(t) e_i(t)$ is the Jacobi field along c, then Jacobi equation can be rewritten as

$$\ddot{J}_i(t) + \sum_{j=1}^{n} R_{ij}(t) J_j(t) = 0, \;\; i = 1, ..., n, \tag{1.9}$$

where $R_{ij}(t) = \langle R(\dot{c}(t), e_j(t))\dot{c}(t), e_i(t)\rangle$. Let $R(t) = (R_{ij}(t))$, which is a symmetric matrix, then $c(t_0)$ is conjugate point if and only if the second order system $\ddot{X}(t) + R(t)X(t)$ with Dirichlet boundary conditions has a nontrivial solution on $t \in [0, t_0]$. It is obvious that $\hat{R}(t) := Tr(R(t))$ is the Ricci curvature in the direction of $\dot{c}$. Set $\hat{R}^+(t) = \frac{1}{2}(\hat{R} + |\hat{R}|)$, then Corollary 1.4 implies that there is no conjugate point alone $[0, a]$ if $\left(\int_0^T \left(t - \frac{t^2}{T}\right)\hat{R}^+ dt\right) < 1$.

Now, if we consider the system (1.8) in the case $n = 1, R = 1$ with the boundary conditions

$$y(0) = 0, \quad \cos(\theta)y(T) + \sin(\theta)\dot{y}(T) = 0, \ \ \theta \in [0, \pi/2]. \tag{1.10}$$

It is well known that the k-th eigenvalue λ_k is the k-th positive solution of the next transcendental equation

$$\tan(\sqrt{\lambda}T) = -\tan(\theta)\sqrt{\lambda}.$$

It is easy to check that if $\theta = 0$, then $\lambda_k = \frac{k^2\pi^2}{T^2}$, and if $\theta = \pi/2$, then $\lambda_k = \frac{\pi^2}{T^2}(k - \frac{1}{2})^2$. For $\theta \in (0, \pi/2)$, it is obvious that $(k - \frac{1}{2})\pi < \sqrt{\lambda_k}T < k\pi$. However, λ_k can only be solved numerically. As an application of the trace formula, we have the following equality, which itself is interesting.

$$\sum_{k\in\mathbb{N}} \frac{1}{\lambda_k} = \frac{3T^2\sin(\theta) + T^3\cos(\theta)}{6(\sin(\theta) + T\cos(\theta))}. \tag{1.11}$$

Obviously, for $\theta = 0$, (1.11) gives the well known identity $\sum\limits_{k\in\mathbb{N}} \frac{1}{k^2} = \frac{\pi^2}{6}$, and for $\theta = \pi/2$, (1.11) gives the identity $\sum\limits_{k\in\mathbb{N}} \frac{1}{\pi^2(k-\frac{1}{2})^2} = \frac{1}{2}$. To the best of our knowledge, for $\theta \in (0, \pi/2)$, we don't know any such kind of formula before on the sum of $\frac{1}{\lambda_k}$. The detailed calculation will be listed in Section 4. Moreover, it is worth to point out that we can compute the value of $\sum \frac{1}{\lambda_k^m}$ for any $m \in \mathbb{N}$ by the trace formula (1.7).

The present paper is organized as follows. In section 2, we give the proof of the Hill-type formula (1.5). Section 3 is devoted to proving the trace formula. Finally, in Section 4, we will give the proof of Corollary 1.4 and identities (1.11).

2 Hill-type formula for Sturm-Liouville systems

In this section, we will give the proof of the Hill-type formula. The following lemma coming from [16, Lemma 3.6] plays a important role.

Lemma 2.1. *Let $f(z)$ be an entire function with zeros at $z_1, z_2, \cdots$ (counting multiplicity). Suppose f satisfied*
i) Exponential bounded condition: for any ϵ, there exist C_ϵ such that

$$|f(z)| \le C_\epsilon \exp(\epsilon|z|),$$

ii) Sum finite condition: $\sum_{n=0}^{\infty} |z_n|^{-1} < \infty$, then

$$f(z) = f(0)\prod_{i=1}^{\infty}(1 - z_n^{-1}z).$$

Since R_1A^{-1} is a trace class operator, by [16, Chapter 3, P33], we have that $\det(I+\lambda R_1A^{-1})$ is an entire function with $|\det(I+\lambda R_1A^{-1})| \le \exp(|\lambda|\cdot\|R_1A^{-1}\|_1)$, where $\|\cdot\|_1$ is the trace norm. Moreover, it satisfied the exponential bounded condition. It is obvious that λ_n is a zero point of $\det(I+\lambda R_1A^{-1})$ if and only if λ_n is an eigenvalue of the system (1.1-1.2). It follows that $\sum\limits_n \frac{1}{|\lambda_n|} < \infty$, where the sum takes for λ_j counting multiplicity. From Lemma 2.1, we have that

$$\det(I+\lambda R_1A^{-1}) = \prod_n (1-\lambda_n^{-1}\lambda). \tag{2.1}$$

To continue, it is easy to verify that y_0 is a solution of (1.1-1.2) with respect to the eigenvalue λ_0 if and only if z_0 is a solution of (1.3) with respect to the same eigenvalue. And it is equivalent to $\gamma_{\lambda_0}(T)z_0(0) \in \Lambda_1$. We have the following observation.

Lemma 2.2. *Suppose that A is nondegenerate, then*

$$dim\ \ker(R_1A^{-1}+1/\lambda_0) = dim\,\gamma_{\lambda_0}(T)\Lambda_0\cap\Lambda_1. \tag{2.2}$$

For the Lagrangian frames Z_i of Λ_i, $i=0,1$, set

$$g(\lambda) = \det(\gamma_\lambda(T)Z_0, Z_1). \tag{2.3}$$

We have

Lemma 2.3. *$g(\lambda)$ is an analytic function and satisfied the exponential bounded condition.*

Proof. The analyticity of γ_λ comes from Krein [14] essentially. For the Taylor expansion, readers are referred to [7, Section 2.2]. Next, we will show that $g(\lambda)$ satisfies the exponential bounded condition. For nonzero λ, let $\mu=\lambda^{1/4}$ and $a(\mu)=diag(\mu I_n,\mu^{-1}I_n)$, we set $\hat\gamma_\lambda(t)=a(\mu)^{-1}\gamma_\lambda(t)$, direct computation shows that

$$\frac{d}{dt}(\hat\gamma_\lambda(T)) = Ja(\mu)B_\lambda(t)a(\mu)\hat\gamma(t).$$

Moreover,

$$a(\mu)B_\lambda(t)a(\mu) = \bar B_\mu(t)+\mu^2\hat B(t)$$

with

$$\bar B_\mu = \begin{pmatrix} 0_n & -P^{-1}Q \\ -Q^TP^{-1} & \mu^{-2}(Q^TP^{-1}Q-R)\end{pmatrix},\quad \hat B = \begin{pmatrix} P^{-1} & 0_n \\ 0_n & -R_1\end{pmatrix}.$$

Let $\bar\gamma_\mu$ be the fundamental solution with respect to $\bar B_\mu$, then

$$\frac{d}{dt}(\bar\gamma_\mu^{-1}\hat\gamma_\lambda(t)) = \mu^2J\bar\gamma_\mu^T\hat B(t)\bar\gamma_\mu\cdot\bar\gamma_\mu^{-1}\hat\gamma_\lambda(t)).$$

Restricting on the region $|\mu|\geqq 1$, it is obvious that $\bar\gamma_\mu$ is bounded, and thus $\bar\gamma_\mu^T\hat B(t)\bar\gamma_\mu$ is bounded. Hence $\|\bar\gamma_\mu^{-1}\hat\gamma_\lambda(T))\| \leqq \exp(C|\mu|^2)$ for some constant C. Consequently

$$\|\gamma_\lambda(T)\| \leqq C_0|\lambda|^{1/2}\exp(C|\lambda|^{1/2}).$$

Finally, notice that $g(\lambda)$ is the finite combination of the finite product of the branches in the matrix, we have the results.

□

By Lemma 2.2, $g(\lambda_0) = 0$ if and only if λ_0 is eigenvalue of (1.1-1.2). That is, $g(\lambda)$ has the same zero points as $\det(I + \lambda R_1 A^{-1})$. Moreover, we have the following lemma.

Lemma 2.4. *Suppose $R_1 > 0$ and λ_0 is a zero point of $g(\lambda)$, then the multiplicity of $g(\lambda)$ at λ_0 is same as the multiplicity of $\det(I + \lambda R_1 A^{-1})$ at λ_0.*

Proof. Suppose the multiplicity of $g(\lambda)$ and $\det(I + \lambda R_1 A^{-1})$ at λ_0 is m_1 and m_2 respectively. Since $R_1 > 0$, the eigenvalue of $R_1 A^{-1}$ is simple and then $m_2 = dim \ker(I + \lambda R_1 A^{-1})$. By Lemma 2.2, we have $m_2 \leqq m_1$. On the other hand, by the techniques of small perturbation (details could be found in [9, Section 4]), we can assume $g(\lambda)$ has m_1 simple zeros near λ_0, and thus $m_1 \leqq m_2$, which implies the result. □

From the above lemmas, we will give the proof of Theorem 1.1.

Proof of Theorem 1.1. We firstly prove the Hill-type formula for the case $R_1 > 0$, by the nondegenerate assumption, 0 is not a zero point of $g(\lambda)$. Please note that both $\det(I + \lambda R_1 A^{-1})$ and $g(\lambda)$ satisfy the exponential bounded conditions and by Lemma 2.4, they have the same zero points with same multiplicities. Next by Lemma 2.1, we have

$$\det(I + \lambda R_1 A^{-1}) = g(0)^{-1} g(\lambda). \tag{2.4}$$

In the general case, choose $\alpha_0 \in \mathbb{R}$ such that $R_1 - \alpha_0 I_n > 0$ and $A + \alpha_0 I_n$ is nondegenerate, then

$$\det(I + \lambda R_1 A^{-1}) = \det[I + \lambda(R_1 - \alpha_0 I_n)(A + \alpha_0 I_n)^{-1}] \cdot \det(I + \alpha_0 A^{-1}) \tag{2.5}$$

By using (2.4) on the two factors of the right hand side of (2.5), we have (2.4) for the general R_1. By taking $\lambda = 1$ we get the desired result (1.5). □

3 Trace formula for Sturm-Liouville systems

In this section, we will prove Theorem 1.3. To do this, we will consider the expansion of the Hill-type formula (2.4). Notice that A^{-1} is a trace class operator, by [16, P47, (5.12)],

$$\det(I + \lambda R_1 A^{-1}) = \exp\Big(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \lambda^m Tr((R_1 A^{-1})^m)\Big). \tag{3.1}$$

Next, we will give the expansion on $g(\lambda)$. Let $V_0 = \Lambda_0 \cap \Lambda_1$, assume $\dim V_0 = k_0$, then $0 \leq k_0 \leq n$. Suppose that $\{d_1, \cdots, d_{k_0}\}$ is an orthonormal basis of V_0, and $\{d_1, \cdots, d_{k_0}, d_{k_0+1}, \cdots, d_n\}$ is an orthonormal basis of Λ_0. Notice that $\mathbb{R}^{2n} = \Lambda_0 \oplus J\Lambda_0$. Therefore, setting $d_{n+j} = Jd_j$, we have $\{d_1, \cdots, d_{2n}\}$ is a basis of $\mathbb{R}^{2n}$ and the matrix $M_1 = (d_1, d_2, \cdots, d_{2n})$ is a symplectic orthogonal matrix. Next, set $V_1 = \Lambda_1 \ominus V_0$, then it is a Lagrangian subspace of $\mathbb{R}^{2n} \ominus (V_0 \oplus JV_0)$. Take an orthonormal basis $\{f_{k_0+1}, \cdots, f_n\}$ of V_1, then $\{d_1, \cdots, d_{k_0}, f_{k_0+1}, \cdots, f_n\}$ is an orthonormal basis of Λ_1.

Let $\{e_k; k = 1, \cdots, 2n\}$ be the standard basis of $(\mathbb{R}^{2n}, \omega_0)$. Obviously $e_{n+k} = Je_k$ and $M_1^T d_j = e_j$ for $1 \leqq j \leqq n$. Notice that $M_1^T(f_{k_0+1}, \cdots, f_n)$ gives a Lagrangian frame of $M_1^T V_1$. By direct computation, for $k_0 + 1 \leqq l \leqq n$ and $1 \leqq j \leqq k_0$,

$$(M_1^T f_l, e_j) = (M_1^T f_l, e_{n+j}) = 0.$$

Rewrite such a frame as $\begin{pmatrix} \tilde{X}_1 \\ \tilde{Y}_1 \end{pmatrix}$, where $\tilde{X}_1$, $\tilde{Y}_1$ are $(n-k_0)\times(n-k_0)$ matrices and $\tilde{Y}_1$ is nonsingular. Let $M_2 = \begin{pmatrix} I_{n-k_0} & -\tilde{X}_1\tilde{Y}_1^{-1} \\ 0_{n-k_0} & I_{n-k_0} \end{pmatrix}$, and

$$M_3 = (I_{2k_0} \diamond M_2)\cdot M_1, \tag{3.2}$$

where $I_{2k_0} \diamond M_2 = \begin{pmatrix} I_{k_0} & 0 & 0 & 0 \\ 0 & I_{n-k_0} & 0 & -\tilde{X}_1\tilde{Y}_1^{-1} \\ 0 & 0 & I_{k_0} & 0 \\ 0 & 0 & 0 & I_{n-k_0} \end{pmatrix}$. Obviously, M_3 is a symplectic orthogonal matrix.

Let $\overline{V}_0 = span\{e_1,\cdots,e_n\}$, $\overline{V}_1 = span\{e_{k_0+1},\cdots,e_{k_0+n}\}$ which are Lagrangian subspaces. Let P_0, P_1 be the orthogonal projections onto $\overline{V}_0$ and $\overline{V}_1$ respectively. For any matrix M on $\mathbb{R}^{2n}$, we always set

$$\mathcal{P}(M) := P_1 M_3 M M_3^{-1} P_0,$$

which is a $n\times n$ matrix with $\mathcal{P}(M)_{i,j} = (M_3MM_3^{-1}e_j, e_{i+k_0})$. In the case $\dim V_0 = 0$ or n, the expression of $\mathcal{P}(M)$ is simple. In fact, rewrite $M_3MM_3^{-1} = \begin{pmatrix} \hat{M}_1 & \hat{M}_2 \\ \hat{M}_3 & \hat{M}_4 \end{pmatrix}$, then $\mathcal{P}(M) = \hat{M}_3$ in the case $\Lambda_0 = \Lambda_1$, and $\mathcal{P}(M) = \hat{M}_1$ in the transversal case $\Lambda_0\cap\Lambda_1 = \{0\}$.

Notice that $\det(M_3) = 1$,

$$g(\lambda) = \det(M_3)\det(\gamma_\lambda(T)Z_0, Z_1) = \det(M_3\gamma_\lambda(T)Z_0, M_3Z_1).$$

Direct computation shows that

$$\det(M_3\gamma_\lambda(T)Z_0, M_3Z_1) = \det(M_3\gamma_\lambda(T)M_3^{-1}M_3Z_0, M_3Z_1) = (-1)^{nk_0}\det(\mathcal{P}(\gamma_\lambda(T)))\det(\tilde{Y}_1).$$

Then

$$g(\lambda)g(0)^{-1} = \det(\mathcal{P}(\gamma_\lambda(T))\cdot\det(\mathcal{P}(\gamma_0(T))^{-1}). \tag{3.3}$$

Set $D = diag(0_n, -R_1)$, then $B_\lambda = B_0 + \lambda D$, from [7], let

$$\hat{D}(t) = \gamma_0^T(t)D(t)\gamma_0(t),$$

and

$$F_k = \int_0^T J\hat{D}(t_1)\int_0^{t_1} J\hat{D}(t_2)\cdots\int_0^{t_{k-1}} J\hat{D}(t_k)dt_k\cdots dt_2dt_1, k\in\mathbb{N}.$$

By Taylor's formula,

$$\gamma_\lambda(T) = \gamma_0(T)(I_{2n} + \lambda F_1 + \cdots + \lambda^k F_k + \cdots),$$

then

$$\mathcal{P}(\gamma_\lambda(T)) = \mathcal{P}(\gamma_0(T)) + \lambda\mathcal{P}(\gamma_0(T)F_1) + \cdots + \lambda^k\mathcal{P}(\gamma_0(T)F_k) + \cdots),$$

where $\mathcal{P}(\gamma_0(T))$ is nonsingular. Set

$$G_k = \mathcal{P}(\gamma_0(T)F_k)\cdot\mathcal{P}(\gamma_0(T))^{-1},\quad for\ \ k\in\mathbb{N}, \tag{3.4}$$

and let $f(\lambda)=\det(I_n+\lambda G_1+\cdots)$, which is an analytic function of λ. It is obvious that $g(\lambda)g(0)^{-1}=f(\lambda)$.

Since $f(\lambda)$ vanishes nowhere near 0, we can write $f(\lambda)=e^{g(\lambda)}$, then by [7, Formula (2.6)] and some direct computation,

$$g^{(m)}(0)/m! = \sum_{k=1}^{m}\frac{(-1)^{k+1}}{k}\Big(\sum_{j_1+\cdots+j_k=m} Tr(G_{j_1}\cdots G_{j_k})\Big). \tag{3.5}$$

Compare the coefficients in (3.1) with (3.5), we have

$$Tr((R_1A^{-1})^m) = m\sum_{k=1}^{m}\frac{(-1)^{k+m}}{k}\Big(\sum_{j_1+\cdots+j_k=m} Tr(G_{j_1}\cdots G_{j_k})\Big). \tag{3.6}$$

This proves Theorem 1.3 because $Tr((R_1A^{-1})^m)=\sum_j\frac{(-1)^m}{\lambda_j^m}$.

Moreover, for the first two terms, we can write it more precisely.

$$\sum_j\frac{1}{\lambda_j} = -Tr(G_1) = -Tr\Big(\mathcal{P}\Big(\gamma_0(T)\cdot J\int_0^T\gamma_0^T(t)D(t)\gamma_0(t)dt\Big)\cdot\mathcal{P}(\gamma_0(T))^{-1}\Big), \tag{3.7}$$

and

$$\begin{aligned}\sum_j\frac{1}{\lambda_j^2} &= Tr(G_1^2)-2Tr(G_2)\\ &= -2Tr\Big(\mathcal{P}\Big(\gamma_0(T)\cdot J\int_0^T\gamma_0^T(t)D(t)\gamma_0(t)J\int_0^s\gamma_0^T(s)D(s)\gamma_0(s)dsdt\Big)\cdot\mathcal{P}(\gamma_0(T))^{-1}\Big)\\ &\quad +Tr\Big(\Big[\mathcal{P}\Big(\gamma_0(T)\cdot J\int_0^T\gamma_0^T(t)D(t)\gamma_0(t)dt\Big)\cdot\mathcal{P}(\gamma_0(T))^{-1}\Big]^2\Big).\end{aligned} \tag{3.8}$$

4 Examples

In this section, we will give some detailed calculation on the trace formula for some special separated boundary value problems for Sturm-Liouville system. At first, we will consider the Dirichlet problem for the system (1.8). Obviously, in this case $A=-\frac{d^2}{dt^2}$, $R_1=-R$. Let $K_n=\begin{pmatrix} I_n & 0_n\\ 0_n & 0_n\end{pmatrix}$, $D=\begin{pmatrix} 0_n & 0_n\\ 0_n & R\end{pmatrix}$. Recall that $\gamma_0(t)$ satisfied $\dot\gamma_0(t)=JK_n\gamma_0(t)$ with $\gamma_0(0)=I_{2n}$. Direct computation shows that $\gamma_0(t)=\begin{pmatrix} I_n & 0_n\\ tI_n & I_n\end{pmatrix}$. It is easy to verify $\gamma_0(t)^{-1}=\begin{pmatrix} I_n & 0_n\\ -tI_n & I_n\end{pmatrix}$. Therefore,

$$J\hat{D}(t)=\gamma_0^{-1}(t)JD(t)\gamma_0(t)=\begin{pmatrix} -tR(t) & -R(t)\\ t^2R(t) & tR(t)\end{pmatrix}.$$

Then

$$J\int_0^T\hat{D}dt=\begin{pmatrix} -\int_0^T tRdt & -\int_0^T Rdt\\ \int_0^T t^2Rdt & \int_0^T tRdt\end{pmatrix},$$

and

$$\gamma_0(T)\cdot J\int_0^T \hat{D}dt = \begin{pmatrix} -\int_0^T tRdt & -\int_0^T Rdt \\ \int_0^T t^2Rdt - T\int_0^T tRdt & \int_0^T tRdt - T\int_0^T Rdt \end{pmatrix}.$$

Obviously, $\mathcal{P}(\gamma_0(T)) = TI_n$, and

$$\mathcal{P}\Big(\gamma_0(T)J\int_0^T \hat{D}dt\Big) = \int_0^T t^2Rdt - T\int_0^T tRdt,$$

thus

$$G_1 = \frac{1}{T}\int_0^T t^2Rdt - \int_0^T tRdt.$$

We have

$$Tr(RA^{-1}) = \sum_j \frac{1}{\lambda_j} = Tr\Big(\int_0^T \Big(t - \frac{t^2}{T}\Big)Rdt\Big).$$

Recall that $R^+ = \frac{1}{2}(R + |R|)$ is nonnegative matrices. let λ_j^+ be the j-th eigenvalue of $\ddot{y} + \lambda R^+ y = 0$ under the Dirichlet boundary conditions, then $\lambda_j > 0$ for $j \in \mathbb{N}$. Similar to the discussion of [7, Theorem 4.12], $\sum_j \frac{1}{\lambda_j} = Tr\Big(\int_0^T \Big(t - \frac{t^2}{T}\Big)R^+dt\Big) < 1$ implies $\lambda_1 > 1$, hence $\frac{d^2}{dt^2} + R^+$ is nondegenerate for $\lambda \in [0,1]$. Since $R^+ \geq R$, we have $\frac{d^2}{dt^2} + R$ is nondegenerate. This proves Corollary 1.4.

At the end of this paper, we will consider (1.8) with the boundary condition (1.10). We choose $d_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $f_2 = \begin{pmatrix} \cos(\theta) \\ -\sin(\theta) \end{pmatrix}$ to be the frame of Λ_0 and Λ_1 separately. Then $M_1 = I_2$, $M_2 = \begin{pmatrix} 1 & \cot(\theta) \\ 0 & 1 \end{pmatrix}$, and consequently $M_3 = M_2$. It is not hard to see, $M_3^{-1} = \begin{pmatrix} 1 & -\cot(\theta) \\ 0 & 1 \end{pmatrix}$. Rewrite $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in short. Direct computation shows that in this case

$$\mathcal{P}(M) = a + c\cot(\theta). \tag{4.1}$$

So we have $\mathcal{P}(\gamma_0(T)) = 1 + T\cot(\theta)$, and easy computations show that

$$\mathcal{P}\Big(\gamma_0(T)J\int_0^T \hat{D}dt\Big) = -\frac{T^2}{2} - \frac{T^3}{6}\cot(\theta). \tag{4.2}$$

We get

$$G_1 = -\frac{3T^2 + T^3\cot(\theta)}{6(1 + T\cot(\theta))} = -\frac{3T^2\sin(\theta) + T^3\cos(\theta)}{6(\sin(\theta) + T\cos(\theta))}. \tag{4.3}$$

By (3.7), we get (1.11). It should be pointed out that maybe the identity (1.11) could obtained by some other method. However, by the trace formula, we can get many other interesting identities directly if we consider different boundary conditions for the Sturm-Liouville system.

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